Advanced Systems Theory

20/01/2026, Tuesday, 11:45 – 13:45

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1 Disturbance decoupling problem (DDP) (10 + 5 + 5 + 5 = 25 pts)

Consider the control system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed, \\ z &= Hx \end{aligned}$$

with

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

(a) Compute $\mathcal{V}^*(\ker H)$.

(b) Show that DDP (with state feedback) is solvable for this system.

(c) Show that $u = Fx$ with $F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$ solves the DDP if and only if $f_3 = -1$.

(d) Does there exist a feedback gain F that solves the DDP and renders $A + BF$ stable?

2 Stabilization algorithm by Bass (2 + 5 + 2 + 1 + 10 = 20 pts)

Suppose that the pair (A, B) is stabilizable.

(a) Show that there exists $\alpha > 0$ such that $-\alpha I - A$ is stable.

(b) Explain why there exists a unique $P \geqslant 0$ such that $(\alpha I + A)P + P(\alpha I + A^T) = 2BB^T$.

(c) Show that $\ker P \subseteq \ker B^T$.

(d) Show that there exists F such that $FP = -B^T$.

(e) Show that $A + BF$ is stable for all F such that $FP = -B^T$.

NOTE: If you cannot answer a subproblem, you may take its result as given for the remaining subproblems.

3 Algebraic Riccati equation (20 pts)

Consider the algebraic Riccati equation $A^T P + PA - PBB^T P + Q = 0$ where P and Q are symmetric matrices. Show that $A - BB^T P$ is stable if P, Q are positive semidefinite and (Q, A) is detectable.

4 Data-driven control (3 + 2 + 8 + 6 + 6 = 25 pts)

Consider the true unknown system

$$\boldsymbol{x}(t+1) = A_{\text{true}}\boldsymbol{x}(t) + B_{\text{true}}\boldsymbol{u}(t)$$

where $t \in \mathbb{Z}_+$, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{u} \in \mathbb{R}^m$. Suppose that the data

$$U_- = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}$$

are harvested from the true system. Define

$$X_- := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \quad \text{and} \quad X_+ := \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}.$$

Also, define the set of all data-generating systems

$$\Sigma := \{(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \mid X_+ = AX_- + BU_-\}.$$

The data (U_-, X) are called *informative for stabilization by state feedback* if there exists K such that $A + BK$ is stable for all $(A, B) \in \Sigma$. We have proven, in the lectures, that the data are informative in this sense if and only if X_- has full row rank and there exists a right inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is stable. Moreover, if these conditions are satisfied then $K = U_-X_-^\dagger$ stabilizes every $(A, B) \in \Sigma$. Checking whether X_- has full row rank is easy. If this condition is satisfied, however, there are infinitely many right inverses in general. As such, verifying the second condition is not straighforward. In this problem, we want to device an algorithm to compute a stabilizing gain from informative data.

Suppose that the data (U_-, X) are informative for stabilization by state feedback. Then, X_- has full row rank. By the rank-nullity theorem, the dimension of $\ker X_-$ is $T - n$. Let $V \in \mathbb{R}^{T\times(T-n)}$ be a matrix whose columns form a basis for $\ker X_-$.

(a) Show that $X_-X_-^T$ is nonsingular.

(b) Let $W = X_-^T(X_-X_-^T)^{-1}$. Show that W is a right inverse of X_-.

(c) Show that $X_-^\dagger$ is a right inverse of X_- if and only if $X_-^\dagger = W + VF$ for some $F \in \mathbb{R}^{(T-n)\times n}$.

(d) Show that (X_+W, X_+V) is stabilizable.

(e) Show that $K = U_-(W + VF)$ stabilizes every $(A, B) \in \Sigma$ if F is such that $X_+W + X_+VF$ is stable.

NOTE: If you cannot answer a subproblem, you may take its result as given for the remaining subproblems.

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10 pts free